Period of an Undamped Pendulum

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Setting up

Suppose we have a ball with mass m that is attached to a mass-less string of length L. It is initially displaced at an angle θ_0 from the vertical. Note that as the pendulum swings, its angle changes with time. Therefore, the angle is a function of time $\theta(t)$.

The differential equation for the undamped pendulum is

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta) = 0$$

It is a second order non-linear homogeneous differential equation which is not very easy to work with. A clever trick to tackle that is to multiply the equation by $\frac{d\theta}{dt}$

$$\frac{d^2\theta}{dt^2} \times \left(\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta)\right) = 0$$

$$\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta)\frac{d\theta}{dt} = 0$$

Notice each term on the left-hand side looks like after we applied the Chain Rule. We can simplify it to become

$$\frac{d\theta}{dt}\left(\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \frac{g}{L}\cos(\theta)\right) = 0$$

Insight: If the derivative of some function is zero, then that function is equal to some constant c.

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \frac{g}{L}\cos(\theta) = c$$

Multiply both sides by 2

$$\left(\frac{d\theta}{dt}\right)^2 - 2\frac{g}{L}\cos(\theta) = 2c = C$$

since $2c$ *is just another arbitrary constant* C

To figure out the constant C, we can assume that the angular velocity during $t = 0$, that is at θ_0, is zero... $\frac{d\theta}{dt}\big|_{\theta=\theta_0} = 0$

$$\left(\frac{d\theta}{dt}\right)^2 - 2\frac{g}{L}\cos(\theta) = 0 - 2\frac{g}{L}\cos(\theta_0) = C$$

$$\therefore C = -2\tfrac{g}{L}\cos(\theta_0)$$

Isolate $\frac{d\theta}{dt}$

$$\frac{d\theta}{dt} = \sqrt{2\frac{g}{L}\cos(\theta) - 2\frac{g}{L}\cos(\theta_0)} \qquad Eq.1$$

Solving the differential equation

From Eq. (1), to not make the equation looks like a cluster mess, we can take $h = 2\frac{g}{L}$. Additionally, we can do Separation of Variables to set up for integration.

$$\frac{d\theta}{\sqrt{h\cos(\theta) - h\cos(\theta_0)}} = dt$$

To make the expression under the square root more 'recognizable, we can use the double angle formula for cosine

$$\cos(\theta) = 1 - 2\sin^2(\frac{\theta}{2})$$

After substituting we get

$$\frac{d\theta}{\sqrt{h - 2h\sin^2(\frac{\theta}{2}) - h + 2h\sin^2(\frac{\theta_0}{2})}} = dt$$

$$\frac{d\theta}{\sqrt{2h\sin^2(\frac{\theta_0}{2}) - 2h\sin^2(\frac{\theta}{2})}} = dt$$

Again, to not make it look messy, we take $H = 2h$ and $z = \sin\left(\frac{\theta_0}{2}\right)$. *(you will see why we don't take the square of it)*

And now we are going to integrate both sides with bounds. A period is the time taken for one oscillation. But since an oscillation is like travelling from $\theta = \theta_0$ to $\theta = 0$ 4 times, we can set the integral to be

$$4\int_0^{\theta_0} \frac{d\theta}{\sqrt{z^2 - \sin^2(\frac{\theta}{2})}} = \int_0^T \sqrt{H}dt$$

$$\int_0^{\theta_0} \frac{d\theta}{\sqrt{z^2 - \sin^2(\frac{\theta}{2})}} = \frac{T\sqrt{H}}{4}$$

To solve the integral, we must introduce a substitution.

Let $sin(\frac{\theta}{2}) = z\sin(\phi)$, $\cos\left(\frac{\theta}{2}\right) = \sqrt{1 - z^2\sin^2(\phi)}$

Differentiated: $\frac{1}{2}\cos\left(\frac{\theta}{2}\right)d\theta = z\cos(\phi)d\phi$

Bounds ($z = \sin\left(\frac{\theta_0}{2}\right)$):

- Lower bound: $\sin\left(\frac{0}{2}\right) = 0 = \sin\left(\frac{\theta_0}{2}\right)\sin(\phi) \rightarrow \sin(\phi) = 0 \rightarrow \phi = 0$
- Upper bound: $\sin\left(\frac{\theta_0}{2}\right) = \sin\left(\frac{\theta_0}{2}\right)\sin(\phi) \rightarrow \sin(\phi) = 1 \rightarrow \phi = \frac{\pi}{2}$

Substituting,

$$\int_0^{\frac{\pi}{2}} \frac{2z\cos(\phi)d\phi}{\sqrt{1 - z^2\sin^2(\phi)}\sqrt{z^2 - z^2\sin^2(\phi)}} = \frac{T\sqrt{H}}{4}$$

$$\int_0^{\frac{\pi}{2}} \frac{2d\phi}{\sqrt{1 - z^2\sin^2(\phi)}} = \frac{T\sqrt{H}}{4}$$

$$T = \frac{8}{\sqrt{H}}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - z^2\sin^2(\phi)}}$$

Since $H = 2h = 4\frac{g}{L}$,

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - z^2\sin^2(\phi)}} = 4\sqrt{\frac{L}{g}}\Omega(z)$$

, where $z = \sin\left(\frac{\theta_0}{2}\right)$

Series Expansion of the elliptical integral

Unfortunately, there is no exact solution to the elliptical integral, $\Omega(z)$. However, we can try to represent it as a series expansion.

Let us try to represent the function inside the integral as an infinite series, specifically Maclaurin series. To do that, we need to take a look at the Maclaurin series of $y = (1-x)^{-\frac{1}{2}}$.

The first few order of derivatives of $y = (1-x)^{-\frac{1}{2}}$ are

- $\frac{dy}{dx} = \frac{1}{2}(1-x)^{-\frac{3}{2}}$
- $\frac{d^2y}{dx^2} = \frac{3}{2}\frac{1}{2}(1-x)^{-\frac{5}{2}}$
- $\frac{d^3y}{dx^3} = \frac{5}{2}\frac{3}{2}\frac{1}{2}(1-x)^{-\frac{7}{2}}$
- $\frac{d^4y}{dx^4} = \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}(1-x)^{-\frac{9}{2}}$

Notice we can see a pattern here. The numerator is just the product of the odd numbers, and the denominator is just product of twos. The $(1-x)$ stays the same but the power decreases by 1 each time.

In general, the kth derivative of y, $\frac{d^ky}{dx^k}$, is

$$\frac{d^ky}{dx^k} = \frac{(2k-1)!!}{2^k}(1-x)^{-\frac{2k+1}{2}},$$

where $(2k-1)!! = (2k-1)(2k-3)(2k-5)\ldots 3 \times 1$

Therefore, the Maclaurin series expansion is

$$\sum_{k=0}^{\infty} \frac{f^k(0)}{k!}x^k = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{k!\, 2^k}x^k$$

Now we can substitute the function inside the integral by taking $x = z^2 \sin^2(\phi)$ to become

$$\Omega(z) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - z^2 \sin^2(\phi)}} = \int_0^{\frac{\pi}{2}} \sum_{k=0}^{\infty} \frac{(2k-1)!!}{k!\ 2^k} z^{2k} \sin^{2k}(\phi) d\phi$$

By the Dominated Convergence Theorem, we can interchange the order of integration and the summation. And since the integral is with respect to the variable ϕ, we can pull out everything that does not depend on it.

$$\Omega(z) = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{k!\ 2^k} z^{2k} \int_0^{\frac{\pi}{2}} \sin^{2k}(\phi) d\phi$$

Now we need to take care of the integral. If you are a mad lad, you might noticed that the integral is actually a Beta Function in disguise. If you are, like me, a retard, let's recall the definition of the Beta Function, $\beta(x, y)$.

Recall

$$\beta(x, y) = 2 \int_0^{\frac{\pi}{2}} \sin^{2x-1}(\phi)\ \cos^{2y-1}(\phi) d\phi = \frac{\Gamma(x)\ \Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(x)$ and $\Gamma(y)$ are the Gamma Function of x and y, respectively. *(If you don't know what the Gamma Function is, get cultured first then come back)*

Our goal now is to represent our integral to look like the beta function and then turn it into the Gamma Function form.

Let start by finding the values for x and y:

- For $\sin(\phi), 2k = 2x - 1 \rightarrow x = \frac{2k+1}{2}$
- For $\cos(\phi), 0 = 2y - 1 \rightarrow y = \frac{1}{2}$ *since* $\cos^0(\phi) = 1$

Substituting we get

$$\int_0^{\frac{\pi}{2}} \sin^{2k}(\phi) d\phi = \frac{1}{2} \frac{\Gamma(\frac{2k+1}{2})\ \Gamma(\frac{1}{2})}{\Gamma(k+1)}$$

Now the expression is much simpler to work with. If you know some facts about the Gamma function, we can simplify it even more.

Here are the facts that we need:

- $\Gamma(k+1) = k!$ for $k \in \mathbb{N}$ *(keep in mind that k are integers)*
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
- $\Gamma(\frac{2k+1}{2}) = \frac{(2k-1)!!}{2^k}\sqrt{\pi}$ *(derived by looking for a pattern)*

With those facts in mind, we get

$$\int_0^{\frac{\pi}{2}} \sin^{2k}(\phi)d\phi = \frac{(2k-1)!!\ \pi}{2^{k+1}\ k!}$$

$$\Omega(z) = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{k!\ 2^k} z^{2k} \times \frac{(2k-1)!!\ \pi}{2^{k+1}\ k!} = \frac{\pi}{2}\sum_{k=0}^{\infty}\left[\frac{(2k-1)!!}{2^k\ k!} z^k\right]^2$$

We can simplify it even more by noticing that $2^k k! = (2k)!!$. Therefore,

$$\Omega(z) = \frac{\pi}{2}\sum_{k=0}^{\infty}\left[\frac{(2k-1)!!}{(2k)!!} z^k\right]^2$$

And thus finally,

$$\boldsymbol{T = 2\pi\sqrt{\frac{L}{g}}\ \sum_{k=0}^{\infty}\left[\frac{(2k-1)!!}{(2k)!!} z^k\right]^2}$$

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1}{2}z^2 + \frac{3}{8}z^4 + \frac{15}{48}z^6 + \dots\right),$$

where $z = \sin\left(\frac{\theta_0}{2}\right)$